Why do forces add vectorially? A forgotten controversy in the foundations of classical mechanics

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Despite the routine treatment it receives today, the parallelogram of forces was the subject of controversy, especially during the 19th century. The controversy concerned the reason why forces compose vectorially. If the parallelogram law is explained statically, then the parallelogram law transcends the laws of dynamics. Alternatively, if the parallelogram law is explained dynamically, then it becomes merely a corollary to Newton's second law. I examine the three most important rival approaches to explaining the parallelogram law and identify their principal strengths, weaknesses, and consequences. The dispute among these approaches ultimately turns on whether the parallelogram law would still have held, if forces had not been governed by Newton's second law.

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I. INTRODUCTION

In today's elementary mechanics textbooks, the parallelogram law for the composition of forces is usually introduced as one aspect of a fundamental property of forces: They can be represented as vectors. The question of why forces can be so represented typically receives scant attention. But it was not always so. Textbooks in the 18th, 19th, and even early 20th centuries frequently devoted considerable attention to various subtle and ingenious derivations of the parallelogram law from allegedly more fundamental principles. Surprising as it might be to us today, the parallelogram law's proper explanation was a matter of lively debate. In the words of a standard reference work from 1880: "The doctrine of the parallelogram of forces has given rise to much controversy, not as to its truth, but as to its derivation…."[1]

The main point at issue was whether the parallelogram of forces is explained by dynamics or by statics. If it is explained dynamically, then (in the words of a 1904 textbook[2]) "statics thus becomes a special case of dynamics, when the forces concerned happen to be in equilibrium." The parallelogram law becomes just a corollary of Newton's second law of motion. In contrast, if the parallelogram law is explained by statics, then statics is autonomous: The laws of statics are separate from and transcend the dynamical laws.

Although this controversy has largely dropped out of sight (and even out of the history of vector analysis[3]), it deserves wider visibility. In this article, I examine the three main approaches to explaining why the parallelogram law holds. I start with the dynamical explanation in Sec. II. Its advocates portrayed it as Newton's own approach and as demonstrating how the parallelogram of forces arises from Newton's second law and the parallelogram of displacements. The dynamical argument depicts the parallelogram of forces as partly rooted in the same geometric principle as the parallelograms of velocities and accelerations. However, the dynamical explanation was criticized for failing to reflect the independence of the parallelogram of forces from Newton's second law and any other dynamical principle.

In Sec. III, I present the most widely accepted static explanation in the mid-19th century, which originated in an 1804 paper by the otherwise obscure French physicist Charles Dominique Marie Blanquet Duchayla.[4] This explanation exploits the "principle of the transmissibility of force," which concerns rigid extended bodies rather than mass points. Consequently, critics of this argument's explanatory power objected that the parallelogram of forces is not confined to forces acting on extended bodies. Duchayla's explanation was also criticized for failing to give a unified explanation of the resultant force's direction and magnitude. Nevertheless, Duchayla's argument had many advocates who praised it for correctly capturing the parallelogram law's independence from dynamics.

In Sec. IV, I examine Siméon Denis Poisson's static explanation,[5] which became increasingly popular in the late 19th century. Like Duchayla's argument, Poisson's explanation belongs to statics. But it avoids the "principle of the transmissibility of force" to which Duchayla appealed, using instead various symmetries, dimensional considerations, and the property that two forces have a unique resultant determined entirely by their magnitudes and directions. Advocates of Poisson's argument saw it not only as giving a unified treatment to direction and magnitude, but also as applicable to a wide range of quantities, explaining why they all compose vectorially despite their diversity.

In Sec. V, I present a summary and discussion. I propose that the dispute among these rival explanations ultimately concerns whether the parallelogram law transcends Newton's second law and the transmissibility principle, or whether it depends on one of them. That is, the dispute turns on whether or not the parallelogram law would still have held, even if Newton's second law or the transmissibility principle had not. If so, then the parallelogram law cannot depend on either of these principles.

These rival explanations of the parallelogram law make an interesting case study of the way that alternative formulations of the same principle can differ in their explanations, the facts they unify, and the relations they regard as coincidental. This dispute also illustrates how questions about one principle's explanatory dependence on another can remain unsettled long after the principles themselves have been firmly established. Students would benefit from observing how textbooks that agree on various familiar principles of mechanics can disagree over which of these principles are responsible for the others.

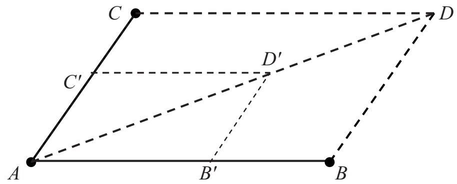

Fig. 1. The dynamical explanation.

II. THE DYNAMICAL EXPLANATION OF THE PARALLELOGRAM LAW

The parallelogram law for the composition of forces first appeared in 1586 in Simon Stevin's *Principles of the Art of Weighing*.[6] (The parallelogram law for the composition of velocities was known to the ancients. It was stated in *Mechanical Problems*,[7] which is traditionally ascribed to Aristotle but may have been written by one of his early followers.) The parallelogram of forces seems to have been widely recognized by Newton's day because both Pierre Varignon and Bernard Lamy stated it in the same year (1687) as Newton published the *Principia*.[8]

Physicists who interpret the parallelogram of forces as arising from dynamics[9] typically characterize the dynamical explanation as having been given (at least loosely) by Newton in deriving Corollary 1 from his second law of motion.[10] The dynamical explanation follows Newton's derivation in applying the second law not only to the resultant force, but individually to each of the two component forces, thereby basing the parallelogram law in the two forces' independence. The dynamical explanation also follows Newton's argument in basing the parallelogram of forces on the parallelogram of accelerations. The dynamical explanation begins by noting that the accelerations produced independently by two forces are in the directions of those forces, are proportional to their magnitudes, and are composed parallelogram-wise. Because the resultant force is in the direction of and proportional to the resultant acceleration, the two forces must likewise compose parallelogram-wise.

Suppose a point particle of mass m is acted on by two forces represented in magnitude and direction by line segments AB and AC with lengths b and c, respectively, and enclosing angle BAC measuring α radians (see Fig. 1, which was often used to illustrate the dynamical explanation[11]). By Newton's second law, the two forces cause accelerations in their respective directions represented by segments AB' and AC' with lengths b/m and c/m, respectively. The resultant acceleration is represented by the diagonal of parallelogram $AB'D'C'$, with length

$$\sqrt{(b/m)^2+(c/m)^2+2(b/m)(c/m)\cos\alpha}. \tag{1}$$

By Newton's second law,[12] the resultant acceleration is associated with a resultant force directed along the diagonal with magnitude

$$m\sqrt{(b/m)^2+(c/m)^2+2(b/m)(c/m)\cos\alpha}, \tag{2}$$

which simplifies to

$$\sqrt{b^2+c^2+2bc\cos\alpha}. \tag{3}$$

Equation (3) is the length of the diagonal of the parallelogram having segments AB and AC as adjacent sides.

Advocates of one derivation of the parallelogram law tend to defend it over others not primarily on the grounds of its being shortest, simplest, or easiest for students to grasp. Rather, the greatest emphasis is typically placed on whether a given derivation is philosophically (or methodologically) correct, that is, whether it correctly identifies which principles are responsible for the parallelogram law. For example, in arguing for the dynamical derivation, William Thomson (Lord Kelvin) and Peter Guthrie Tait emphasized that "we believe it contains the most philosophical foundation"[13] for physics. In contrast, the dynamical argument's critics generally charge it not with being unnecessarily complicated or difficult to follow, but rather with being "unnatural and a defect in method"[14] (in the words of an American advocate of Poisson's explanation, given in Sec. IV). Although the statical derivations are arguably more complicated than the dynamical derivation, their advocates (as the following sections will illustrate) tend to advance them on philosophical rather than pedagogical grounds–in particular, as reflecting the actual relations of explanatory dependence among various physical principles.

A common objection to the dynamical explanation is that the law determining the resultant of two forces is fundamentally nothing more than the law determining which three forces combine to yield zero force, because forces F and G combine to yield force R if and only if F, G, and a force equal and opposite to R combine to yield zero force.[15] Furthermore, the law determining which forces combine to yield zero force (that is, which forces are in equilibrium) is independent of what would happen in a non-equilibrium case. That is, the law determining which forces are in equilibrium is not based on the precise connection between force and motion. Therefore, notions such as motion and mass should not enter into an explanation of the parallelogram law.

However, these notions figure prominently in the dynamical argument. It turns on the relation between force and motion. But because in equilibrium there is no effect on motion, and because the law of the composition of forces is equivalent to the law specifying the conditions of equilibrium, the particular motion that would ensue under non-equilibrium must be irrelevant to the parallelogram law's explanation. Its irrelevance is demonstrated (according to this objection to the dynamical explanation) by the fact that the mass m is introduced into the dynamical argument only to cancel out in the step from Eq. (2) to Eq. (3). According to this objection, m serves in the dynamical argument as an artificial device like an arbitrary auxiliary line drawn to facilitate the proof of a geometric theorem, but having nothing to do with why the theorem holds.

Critics of the dynamical argument often cite William Whewell as having pressed the case against the dynamical explanation in the first half of the 19th century. (Whewell, who rose from undergraduate to Master at Trinity College, Cambridge, was an immensely influential polymath who not only won a gold medal from the Royal Society for his research on ocean tides, but also wrote prolifically on a wide range of subjects, including mechanics, logic, and the history of science. He also coined the terms "scientist" and "physicist.") For example, James Challis (the Plumian Professor of Astronomy at Cambridge, most famous today for having narrowly missed discovering Neptune in 1846) praised Whewell for having done "away with the illogical method of proving the parallelogram of forces by means of bodies in *motion*, which had previously been adopted in English works" (ital-

ics in the original).[16] Whewell characterized the parallelogram of forces as "independent of any observed laws of motion."[17] He wrote:

> "The composition of motion… has constantly been confounded with the composition of force. But… it is quite necessary for us to keep the two subjects distinct…. The conditions of equilibrium of two forces on a lever, or of three forces at a point, can be established without any reference whatever to any motions which the forces might, under *other* circumstances, produce.… To prove such propositions by any other course, would be to support truth by extraneous and inconclusive reasons…"[18]

Many physicists have agreed with Whewell that the dynamical argument cannot explain why the parallelogram of forces holds because the parallelogram law depends exclusively on statics. For instance, William H. Macaulay (the English mathematician and physicist after whom "Macaulay's method" for determining the deflection of beams is named) emphasized that this objection to the dynamical explanation is philosophical rather than pedagogical:

> "[T]o make [the parallelogram of forces]… dependent on a theory of mass, as appears to be usual in modern text-books, somewhat grates upon one's sense of logical order."[19]

In contrast, the dynamical explanation's defenders frequently praised it for unifying the parallelogram of forces with the parallelograms of accelerations and velocities, giving them a common origin in the parallelogram of displacements. For example, after characterizing Whewell's proposed static explanation as "forced and unnatural," one proponent of the dynamical explanation wrote:

> "With regard to mechanics itself, the extended and comprehensive view of this subject would embrace what are usually termed the parallelogram of forces and the composition of motion (or the 'parallelogram of velocities,' as it is sometimes called) in the same fundamental idea. The separation of these two things—which are in reality but one—is justly censured by Lagrange, as depriving them of their 'evidence and simplicity'."[20]

In a remark widely endorsed[21] as capturing an important feature of the dynamical explanation, Lagrange wrote that the dynamical explanation "has the advantage of demonstrating clearly why the composition of forces necessarily follows the same laws as the one of velocity."[22] As I will discuss in Sec. IV, a similar unification argument is used to defend Poisson's static explanation of the parallelogram law.

III. DUCHAYLA'S STATIC EXPLANATION

The explanation of the parallelogram law usually given[23] in mid-19th century textbooks[24] originated with Duchayla in 1804.[25] Charles Dominique Marie Blanquet Duchayla was a student at the Ecole Polytechnique in 1795–1796 (3 years before Poisson). In 1806 he assisted Arago in his experiments on the velocity of light. He became a naval engineer

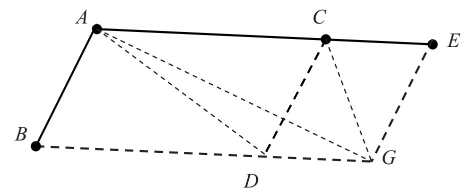

Fig. 2. Duchayla's explanation of the rule for the inductive step.

and later taught at the academies in Turin, Montpelier, and Aix. His 1804 paper was apparently his only scientific publication.[26]

Although Duchayla's derivation is subtle and complex, 19th century textbooks commonly presented it in full. For example, an 1893 applied mechanics text intended for engineering students devoted three pages to giving it in detail.[27] To illustrate the argument's steps, I will use the same figures that many textbooks employed. Duchayla's explanation proceeds like a proof by mathematical induction. I will start (as was customary) with the rule for the argument's inductive step, giving Duchayla's account of why the rule holds. Duchayla appealed to the "principle of the transmissibility of force." Then I will give Duchayla's argument by induction, whereby he explained why the parallelogram law gives the resultant force's direction. Finally (continuing to follow the standard order of presentation), I will show how Duchayla used the parallelogram law for the resultant's direction to explain why the parallelogram law also holds for the resultant's magnitude.

Let us begin with the rule for the derivation's inductive step: If forces P and Q, acting together at a point, result in a force directed along the diagonal of the parallelogram representing the two forces, and if the same applies to forces P and R acting together at the same point, with R acting in Q's direction, then the same applies to P acting together with the resultant of Q and R. Here is Duchayla's account of why this rule holds. Let P be represented by segment AB (see Fig. 2) and be directed toward B. Grant that the resultant of Q and R (each directed toward E) is in their common direction and is equal in magnitude to the sum of their magnitudes, and let it be represented by segment AE, with Q represented by AC, so that segment CE is the proper length and direction to represent R, except that R is actually applied at A rather than C. Now Duchayla appeals to the principle of the transmissibility of force, which says that when a force acts on a body, the result is the same whatever the point (rigidly connected to the body) at which it is applied, provided that the line through that point and the force's actual point of application lies along the force's direction. By this principle, although R is applied at A, its effect is the same if it is applied at C because AC is in the force's direction. With the parallelograms in Fig. 2 forming a rigid body, the three forces can be applied to other points along their lines of action without changing their resultant. By assumption, the resultant of P and Q acts along the diagonal AD, so it can be applied at D. It can then be resolved into P and Q acting at D. Q's direction is along DG, so Q can be transferred to G. P's direction is along CD, so P can be transferred to C, where it meets R. By assumption, their resultant acts along diagonal CG, so it can be transferred to G, meeting Q. Finally, by the converse of the transmissibility principle, AG must be along the line of action of the composition of P with the resultant of Q and R.[28]

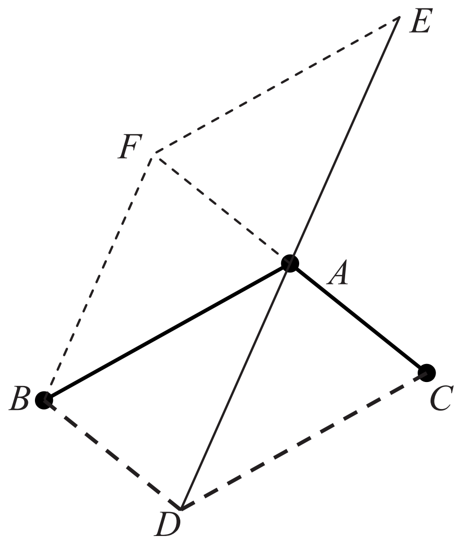

Fig. 3. Duchayla's explanation of the resultant's direction.

We see that Duchayla used the transmissibility principle (and its converse) to explain why the rule for the inductive step holds. Duchayla then appealed to symmetry to explain why the resultant of two equal forces acting at the same point is represented by an arrow from that point in the same plane as the arrows representing the two forces and bisecting the angle between them.[29] This conclusion was the first step in Duchayla's inductive argument. Duchayla then considered a given force *F*, another force *G* with *F*'s magnitude but in a different direction, and a third force *H* identical to *G*, all acting at the same point. Because the resultant of *F* and *G* lies along the corresponding parallelogram's diagonal, and likewise for *F* and *H*, the inductive rule shows that the resultant of *F* composed with a force of twice *F*'s magnitude (the resultant of *G* and *H*) points along the corresponding parallelogram's diagonal. This argument can be repeated indefinitely (initially with *F*, the force of twice *F*'s magnitude, and *H*, but then with progressively larger multiples of *F*) to extend the explanation to any two commensurable forces, that is, to nF composed with mF for any positive integers n and m. Because any two incommensurable forces can be approximated to any desired degree by a pair of commensurable forces, the explanation can be extended to any two forces.

In this way Duchayla explained why the parallelogram law holds with regard to the resultant's direction. The final step of Duchayla's argument aims to explain why it also holds for the resultant's magnitude. Let *AB* and *AC* (see Fig. 3) represent two forces. Their resultant was just shown to point in the direction of *AD* (the parallelogram's diagonal). Let *AE* extend *AD* in the opposite direction, its length representing the resultant force's magnitude. Duchayla showed that the parallelogram's diagonal represents the resultant's magnitude by showing *AD* equal to *AE*. The forces represented by *AB*, *AC*, and *AE* balance. Construct parallelogram *AEFB*. The resultant of the forces represented by *AB* and *AE* was just shown to point along the diagonal *AF*. Hence *AF* must lie along the same straight line as *AC*. Because they are opposite sides of parallelogram *ACDB*, *AC* and *BD* are parallel. Because *AF* and *AC* lie along the same line, *AF* is parallel to *BD*. Because they are opposite sides of parallelogram *AEFB*, *AE* and *BF* are parallel. Because *AE* and *AD* lie along the same line, *AD* is parallel to *BF*. Because *AF*-*BD* and *AD*-*BF* are pairs of parallels, *AFBD* is a parallelogram. Its opposite sides *AD* and *BF* are equal. As opposite sides of parallelogram *AEFB*, *AE* and *BF* are equal. Therefore, *AD* and *AE* (both equal to *BF*) are equal, as Duchayla set out to show.[30]

So ends Duchayla's explanation. Despite its length, mid-19th century texts frequently praised it as "very simple and beautiful."[31] However, toward the end of the 19th century, Duchayla's argument increasingly came under heavy criticism. Although all agreed that the parallelogram law can be deduced by Duchayla's procedure, his argument was increasingly regarded as failing to explain why the law holds.[32] Two specific deficiencies in Duchayla's argument were commonly cited.

Many physicists contended that the parallelogram of forces does not actually depend on one of the key premises in Duchayla's argument: The principle of the transmissibility of force. As Macaulay put it (echoing Whewell's characterization of Newton's second law as "extraneous" to the parallelogram law):

> "Duchayla's proof… held its own in text-books for many years… in spite of the extraneous feature which the appeal to the transmissibility of forces intrudes. Logically the best form of proof seems to be one on the lines adopted by Laplace."[33]

(I shall turn to that form of explanation in Sec. IV.) The Cambridge logician, mathematician, and economist W.E. Johnson (who endorsed the familiar criticism of the dynamical explanation as introducing "kinetic ideas which are really nowhere again used in statics") explained more fully why the transmissibility principle is "extraneous" to the parallelogram law:

> "To base the fundamental principle of the equilibrium of a *particle* on the 'transmissibility of force', and thus to introduce the conception of a *rigid body*, is certainly the reverse of logical procedure."[34]

Whereas Macaulay and Johnson contended that the parallelogram law, as a fact about point particles, cannot rest on a fact about rigid extended bodies, proponents of Duchayla's explanation contended that statics is concerned fundamentally with rigid extended bodies rather than material points. As Challis put it, "Statics is restricted to the equilibrium of *rigid* bodies."[35]

The emphasis that Macaulay and Johnson placed on correct "logical procedure" is similar to the philosophical grounds on which the dynamical argument is generally evaluated. Although Thomson and Tait differed from Macaulay and Johnson in favoring the dynamical explanation, they joined Macaulay and Johnson in criticizing Duchayla's argument for appealing to the transmissibility principle. With Duchayla's explanation in mind, they criticized derivations of the parallelogram law involving

> "… the introduction of various unnecessary Dynamical axioms, more or less obvious, but in reality included in or dependent upon Newton's laws of motion."[36]

Thomson and Tait criticized these derivations not for being difficult to follow, but primarily for misrepresenting which principles really depend on others. According to Thomson and Tait, the parallelogram law and the transmissibility of force do not reside in an autonomous statics, but arise together with dynamics from Newton's laws.[37]

The second deficiency that critics identified in Duchayla's proposed explanation is that it incorrectly represents the way in which the parallelogram law's applicability for direction relates to its applicability for magnitude. Duchayla's argument depicts these two aspects of the parallelogram law as having different explanations. Duchayla's final argument appeals to the parallelogram law for direction to explain why the parallelogram law also holds for magnitudes. But rather than one depending on the other, these two aspects of the parallelogram law must (according to this objection) arise in the same way from the same, more fundamental principles.

This objection is pressed, for instance, by the Caius College, Cambridge mathematician (and later bishop of Carlisle) Harvey Goodwin. Decrying Duchayla's proof as "artificial," he focused on its final step (in which the parallelogram law for the resultant's direction is used to explain why the parallelogram law holds for the resultant's magnitude):

> "[T]he extreme simplicity of this part of the proof shews how intimate the connexion must be between the two parts of the proposition, a connexion which I think we should not have been led to expect from anything occurring in the proof itself… [From the proof of the parallelogram law for direction] there is not a shadow of a hint that… the law will hold as respects magnitude: so that a very remarkable proposition is proved by a mere artifice without apparently the least reason in the nature of things why we should anticipate the result."[38]

IV. POISSON'S STATIC EXPLANATION

Poisson's explanation perfected a strategy developed earlier by Foncenex, D'Alembert, Daniel Bernoulli, Laplace, and Cauchy.[39] It became the chief rival to the dynamical explanation toward the end of the 19th century.[40] It is not susceptible to either of the two major objections to Duchayla's explanation: it appeals to symmetries and dimensional considerations in place of the transmissibility principle, and it treats magnitude and direction together. Textbooks of the period frequently devote several pages to giving the explanation in full, often illustrating it with the same figures I shall use.[41]

The central step of Poisson's derivation explains why the parallelogram law gives the resultant of two forces P_1 and P_2, each of magnitude P, acting in directions subtending angle $2x$ (less than π). The explanation begins with the premise that the resultant of P_1 and P_2 must not only (by symmetry) be coplanar with P_1 and P_2, but also be unchanged under an interchange of P_1 and P_2 and thus must bisect the angle between them.[42] Poisson then appealed to the premise that the resultant must reflect only the two forces' magnitudes and directions. Hence, its magnitude R must be some function f of P and x alone. Because $R=f(P,x)$ must hold in any system of units for R, P, and x, $f(P,x)$ must equal $P^\alpha g(x)$ for some dimensionless α and

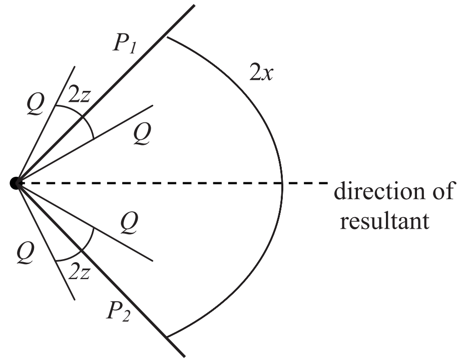

Fig. 4. Poisson's explanation for the special case of two equal forces.

some dimensionless function g.[43] Because R and $P^\alpha g(x)$ must both have dimensions of force and P has dimensions of force, it follows that $\alpha=1$, and hence $R=Pg(x)$.

Poisson had now reduced the problem to explaining why g is in accordance with the parallelogram law. To create another instance of two equal forces acting in different directions, let P_1 be the resultant of two forces, each of magnitude Q, subtending angle $2z$ (see Fig. 4). Hence $P=Qg(z)$. Because $R=Pg(x)$, it follows that $R=Qg(z)g(x)$. By the composition law's rotational symmetry, these two forces rotated through the same angle have P_2 as their resultant (see Fig. 4). The resultant of the four Q forces is the resultant of P_1 and P_2, with magnitude R. The two inner Q forces in Fig. 4 create another instance of the same problem: Their resultant bisects the angle $2(x-z)$ between them, and therefore it points in the same direction as the resultant of P_1 and P_2 and has magnitude $Qg(x-z)$. Likewise, the resultant of the two outer Q forces is in the same direction, with magnitude $Qg(x+z)$.

Poisson then invoked the principle that if two forces point in the same direction, their resultant's magnitude is the arithmetic sum of their magnitudes. Therefore, as the magnitude of the four Q forces' resultant, R equals the sum of $Qg(x-z)$ and $Qg(x+z)$. Because $R=Qg(z)g(x)$, we have

$$Qg(z)g(x) = Qg(x-z) + Qg(x+z), \tag{4}$$

and thus

$$g(z)g(x) = g(x-z) + g(x+z). \tag{5}$$

The solution to this functional equation is

$$g(z) = 2\cos az, \tag{6}$$

for arbitrary a. Therefore,

$$R = Pg(z) = 2P\cos ax. \tag{7}$$

To determine a, Poisson assumed that if two forces are equal and opposite, then they have zero resultant.[44] Although Poisson presented this premise as an independent assumption, it follows from premises to which he has already appealed. If equal and opposite forces are rotated by a half-circle around an axis perpendicular to their common line of action, then by rotational symmetry, the resultant is rotated by the same amount. But because the two forces have merely swapped places, the resultant must be unchanged because the resultant depends only on the forces' magnitudes and directions. The

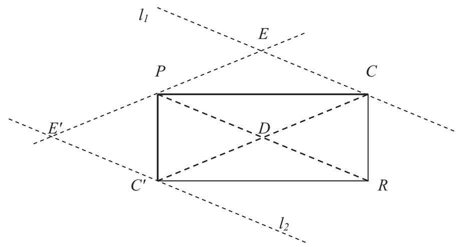

Fig. 5. Poisson's explanation for the special case of two unequal, orthogonal forces.

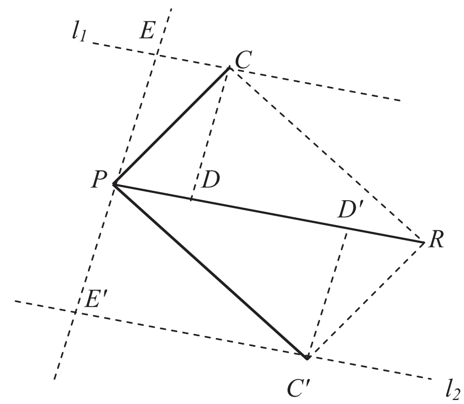

Fig. 6. Completing Poisson's explanation.

only resultant unchanged under such rotation is a zero resultant.

From Eq. (7) for $R=0$ when $x=\pi/2$ (remember that the angle subtended by the two forces is $2x$, see Fig. 4), it follows that a must be an odd integer. But if $a=3$, for example, then $\cos ax=0$ for $x=\pi/6$, and so by Eq. (7), $R=0$. This zero resultant for forces subtending an angle $2x=\pi/3$ is disallowed by Poisson's premise that if two forces have zero resultant, then they must be equal and opposite. Again, although Poisson presented this premise as an independent assumption, it follows from premises to which he had already appealed. If forces A and B have zero resultant, then the resultant of A, B, and some third force is the third force. But if the third force is equal and opposite to B, then by the previous result, it and B have zero resultant. Therefore, the three forces' resultant must be A, which contradicts the resultant's uniqueness unless A and the third force are the same, that is, unless A is equal and opposite to B.[45]

By this reasoning, a cannot be an odd integer greater than 1. Hence, $a=1$, and thus by Eq. (7), $R=2P\cos x$, which is the length of the diagonal of the rhombus with side P, angle $2x$. Thus, Poisson explained why the parallelogram law works in the special case of two equal forces.

He then extended the explanation to unequal, orthogonal forces. Suppose they are applied at P (see Fig. 5) and represented by PC and PC'. Complete the rectangle $CPC'R$. Use the previous result to decompose each of these two forces into two equal forces: one along the rectangle's diagonal, the other canceling its partner from the other force's decomposition. Draw diagonals meeting at D and draw line $l_1(l_2)$ through $C(C')$ parallel to PR. Draw a line through P parallel to CC', and let $E(E')$ be its point of intersection with $l_1(l_2)$. Because PR bisects CC', $C'D=DC$, and thus $PE'=PE$ (because l_1 and l_2 are parallel to PR). Treat PE' and PE as representing hypothetical equal and opposite forces acting at P, directed toward E' and E, respectively. Because they have no resultant, they can be joined with the PC and PC' forces without changing the resultant. $EPDC(E'PDC')$ is a rhombus, and therefore (by the result just shown) $PC(PC')$ represents the resultant of the forces represented by $PE(PE')$ and PD. Consequently, the resultant of the PC and PC' forces is the force represented by the resultant of PD, PD (again), PE, and PE'. Hence it is twice the force represented by PD, which is the force represented by the diagonal PR.

Finally, Poisson employed a similar strategy to extend this result to any two forces, completing his explanation of the parallelogram law. Suppose the two forces are applied at P (see Fig. 6) and represented by PC and PC'. Complete the parallelogram $PCRC'$. Draw $l_1(l_2)$ through $C(C')$ parallel to diagonal PR, draw a line through P perpendicular to PR, and let $E(E')$ be its point of intersection with $l_1(l_2)$. Draw a line through $C(C')$ perpendicular to PR, and let $D(D')$ be its point of intersection with PR. Triangles CRD and $C'PD'$ are congruent because they have two pairs of angles and a pair of corresponding non-included sides congruent: Angles CDR and $C'D'P$ are congruent (both are right angles), $PC'=CR$ (they are opposite sides of a parallelogram), and angles CRD and $C'PD'$ are congruent (as alternate interior angles). Therefore, $CD=C'D'$, and thus $PE=PE'$. Treat PE and PE' as representing hypothetical equal and opposite forces acting at P, directed toward E and E'. Because they have no resultant, they can be joined with the PC and PC' forces without changing the resultant. $CEPD(C'E'PD')$ is a rectangle, and thus (by the result just shown) $PC(PC')$ represents the resultant of the forces represented by $PE(PE')$ and $PD(PD')$. Consequently, the resultant of the PC and PC' forces is the force represented by the resultant of PD, PD', PE, and PE', and therefore is the resultant of the forces represented by PD and PD'. Because triangles CRD and $C'PD'$ are congruent, $DR=PD'$, and therefore the resultant of the PD and PD' forces is the force represented by PR, thus explaining the parallelogram law.

Unlike Duchayla's explanation, Poisson's explanation unites the parallelogram law for the resultant's magnitude with the law for the resultant's direction. Except in the explanation's first stage, where the resultant's direction follows immediately from symmetry, the resultant's direction and magnitude are treated together throughout the argument. Unlike the dynamical explanation, Poisson's explanation portrays the parallelogram law as independent of the relation between force and motion. Little about force in particular is involved anywhere in Poisson's explanation, suggesting an objection expressed, though significantly not endorsed, by the British mathematician Augustus De Morgan:

> "[M]any have been puzzled by finding that the thing which, by its very definition, tends to produce motion, is reasoned on… under a compact that any introduction of the idea of motion would be out of place. The statical proofs… seem to be *all geometry and no physics*." (italics in original)[46]

Proponents of Poisson's explanation seize upon this feature as favoring his explanation over its rivals. It unifies the parallelogram of forces not only with the parallelograms of dis-

placements, velocities, and accelerations, but also with the parallelograms of gravitational fields, bulk magnetizations, electric current densities through three-dimensional conductors, energy flux densities, water flux densities through soils, sound and light flux densities through three-dimensional media, entropy fluxes, heat flows—all quantities that are composed by vector addition. If each of these various parallelogram laws is explained by a Poisson-style argument, then it is no coincidence that these quantities, despite their physical differences, all compose in the same way. Poisson's explanation shows why any quantity composes parallelogram-wise as long as it exhibits certain features (namely, the symmetries and other premises in Poisson's explanation). As Maxwell wrote:

> "[T]he proof which Poisson gives of the 'parallelogram of forces' is applicable to the composition of any quantities such that turning them end for end is equivalent to a reversal of their sign."[47]

V. DISCUSSION

When today's textbooks introduce the parallelogram of forces, the discussion is typically confined to some empirical evidence for the law and some examples of its application in solving problems. That forces are vectors is generally treated simply as a fundamental principle. Nowhere is it suggested that there are several different ways in which the law might perhaps be explained, or that anyone ever seriously tried to explain why it holds, much less that its proper explanation was ever the subject of sharp debate. I believe that it would interest some students to learn not only about the various accounts of why the parallelogram law holds, but also about the various arguments given for and against these proposals.

These accounts and arguments illustrate several ways in which rival explanations of the same law can disagree without making different empirical predictions. The dynamical explanation differs from the two static explanations in portraying the parallelogram law as arising from Newton's second law. Duchayla's explanation differs from its two rivals in describing the parallelogram law as depending on the transmissibility principle and therefore in taking mechanics as fundamentally about rigid extended bodies. Poisson's proposal is distinctive not only in taking various symmetries and dimensional considerations as explaining why forces compose vectorially, but also in identifying features that forces share with various other directed quantities and that are responsible for their all composing vectorially.[48]

These various accounts of the reason why forces add vectorially are genuine rivals. They are incompatible explanations. Advocates of the dynamical explanation believe that laws of statics, such as the parallelogram law, are corollaries of dynamical laws. In contrast, advocates of a static explanation do not believe that the parallelogram law may also be explained dynamically. Rather, they believe (in Macaulay's words) that the static picture "gives a better view of its true position," whereas the dynamical picture "grates upon one's sense of logical order."[49] The idea that contrary to the dynamical account, Newton's second law is "out of place" (as De Morgan said[50]) in an explanation of the parallelogram law is nicely expressed by the English mathematician Olinthus Gregory:

> "It may be proper to remark here that the Composition and Resolution of *forces*, and the similar Composition and Resolution of *motions*, are completely distinct objects of enquiry…. Some authors have inferred from their demonstrations of the latter problem, the truth of the former: but this cannot well be admissible, because wherever statical equilibrium obtains there can be no motion, and of course the principle on which the inference is grounded [namely, Newton's second law] is foreign to the nature of the thing to be proved."[51]

In this view, the parallelogram of displacements and the parallelogram of forces are properly unified not though Newton's second law, but rather through forces and displacements sharing the same symmetries.

What is at stake in this disagreement? What would make the parallelogram of forces have a static rather than a dynamic explanation? That the parallelogram of forces is independent of Newton's second law was commonly expressed by advocates of a static explanation as the idea that forces would still have composed parallelogram-wise even if force had not been governed by Newton's second law–that is, even if force had stood in a different relation to motion. For example, in 1726, Daniel Bernoulli made a pioneering attempt at giving a static explanation of the parallelogram of forces. To motivate it, he suggested that a wide range of alternatives to Newton's second law might have held, but did not. Among these "counterfactual" alternatives are that the resultant force is proportional to the resultant acceleration's square root, or to its cube root, or to its square. But even if one of these counterfactual possibilities had been the case, he maintained, the parallelogram of forces would still have held.[52]

Whether the parallelogram of forces would still have held, even if dynamics had not been governed by Newton's second law, seems to be the main fact in dispute between advocates of static and dynamic explanations of the parallelogram law. It was identified by the English mathematician James Cockle, for instance, as the crucial respect in which the static explanation developed by Bernoulli and Poisson contrasts with the dynamic explanation favored by Thomson and Tait:

> "[W]e may seek to arrive at the composition of pressures, independently of the second law of motion, by processes which are valid whether that law be a law of nature or not, and which would be valid even if we had not any conception of motion, and which, indeed, do not render it necessary to consider whether pressure does or does not tend to produce motion."[53]

Likewise, the French mathematician and physicist Louis Poinsot, in offering a static explanation of the parallelogram law that (like Duchayla's) appeals to the principle of the transmissibility of force, maintained that the laws of statics would still have held even if Newton's second law had not held:

> "[I]n Statics properly so called it is not necessary to know the actual effect of forces upon matter… to ascertain, for instance, if a double force pro-

duces upon the same body a double velocity, or if the same force applied to a body of double the mass produces but half the velocity, et cetera. [In statics,] whatever the action of forces upon bodies may be, be they [the forces] proportional or not to their sensible effects, still the truths which we are about to expound will remain no less the same…"[54]

The parallelogram law's invariance under certain circumstances is likewise the main point over which defenders of Duchayla's and Poisson's explanations disagree. Advocates of Poisson's account hold that contrary to Duchayla's account, the parallelogram law would still have held even if the principle of the transmissibility of force had not been true. For example, in the preface to his 1845 textbook, the English physicist and mathematician Samuel Earnshaw (who first proved Earnshaw's theorem in electrostatics) wrote that although he had decided to introduce the parallelogram law in his book by proving it "after Duchayla's method, by reason of its simplicity,"[55] this proof fails to explain why the parallelogram law holds because of the same defect that Macaulay and Johnson would later emphasize: Its appeal to the transmissibility principle. Like Macaulay and Johnson, Earnshaw regarded Poisson's argument rather than Duchayla's as explanatory:

> "I think it necessary here to inform the reader that, as [Duchayla's] method is inapplicable when the forces act upon a single[56] particle of matter (as a particle of a fluid medium on the hypothesis of finite intervals), on account of its assuming the transmissibility of the forces to other points than that on which they act, I have, in an Appendix, given [Poisson's proof]…."[55]

Earnshaw distinguished one proof's pedagogical advantages from another proof's philosophical virtues. He then clarified why the transmissibility principle cannot help to explain the parallelogram law. Here counterfactual considerations enter:

> "[Duchayla's] method… can never be exclusively adopted in a treatise which professes to take a more philosophical view of the subject; for, were the transmissibility of force *not* true in fact, the law of the composition of forces acting on a point would still be true; it is evident, therefore, that to make the truth of the former an essential step in the proof of the latter, is erroneous in principle."[55]

According to Earnshaw, the key point is that the parallelogram law cannot be explained by the transmissibility principle because the parallelogram law would still have held even if the transmissibility principle had not been true.[57]

I conclude that facts expressed in such counterfactual terms, which capture the dependence of certain laws on others, are the principal point on which advocates of the three rival explanations disagree. Accordingly, advocates of these rival explanations all agree on one point: That (in Eddington's words) "the contemplation in natural science of a wider domain than the actual leads to a far better understanding of the actual."[58]

[1] Library of Universal Knowledge, A Reprint of the Last *Edinburgh and London Edition of Chambers's Encyclopædia* (American Book Exchange, New York, 1880), Vol. IV, pp. 208–209.

[2] J. Cox, *Mechanics* (Cambridge U. P., Cambridge, 1904), p. 68.

[3] The standard reference is M. J. Crowe, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System* (University of Notre Dame Press, Notre Dame, IN, 1967).

[4] C. D. M. B. Duchayla, "Demonstration du parallèlogramme des forces," Bulletin des Sciences par la Société Philomathique de Paris **4**, 242–243 (1804).

[5] S. D. Poisson, *Traité de Mécanique* (Courcier, Paris, 1811), Vol. 1, pp. 11–19; *A Treatise of Mechanics*, 2nd ed. (Bachelier, Paris, 1833) Vol. 1, translated by H. H. Harte (Longmans, London, 1842), pp. 36–42.

[6] S. Stevin, in *The Principal Works of Simon Stevin*, edited by E. J. Dijksterhuis (C. V. Swets & Zeitlinger, Amsterdam, 1955), Vol. 1, pp. 174–179.

[7] Aristotle, *Minor Works, Loeb Classical Library No. 307*, translated by W. S. Hett (Harvard University Press, Cambridge, 1980), pp. 337–339.

[8] I. Newton, *The Principia: Mathematical Principles of Natural Philosophy*, translated by I. B. Cohen and A. Whitman (University of California Press, Berkeley, CA, 1999) See R. Dugas, *A History of Mechanics* (Dover, New York, 1988); P. Duhem, *The Origins of Statics* (Kluwer, Dordrecht, 1991).

[9] Among the texts endorsing the dynamical explanation are J. Horsley, *A Short and General Account of the Most Necessary and Fundamental Principles of Natural Philosophy* (Andrew Stalker, Glasgow, 1743), p. 14; T. Rutherford, *A System of Natural Philosophy* (J. Bentham, Cambridge, 1748), Vol. 1, pp. 16–17; W. H. C. Bartlett, *Elements of Natural Philosophy*, Vol. 1 (A. S. Barnes, New York, 1850), p. 109; P. G. Tait, *Lectures on Some Recent Advances in Physical Science*, 3rd ed. (Macmillan, London, 1885), p. 356; W. Thomson and P. G. Tait, *Treatise on Natural Philosophy* (Cambridge U. P., Cambridge, 1888), Vol. 1, pp. 244–245; J. B. Lock, *Elementary Dynamics* (Macmillan, London, 1891), pp. 104–105; Oliver Lodge, *Elementary Mechanics*, rev. ed. (W. & R. Chambers, London and Edinburgh, 1890), p. 96; S. L. Loney, *The Elements of Statics and Dynamics*, Vol. 2 (Cambridge U. P., Cambridge, 1891), p. 60; Ref. 2, p. 158.

[10] I. Newton, Ref. 8, pp. 417–418.

[11] See, for instance, W. H. C. Bartlett, Ref. 9, and S. L. Loney, Ref. 9. The following version of the dynamical explanation generally follows the version appearing in many texts but makes more explicit the roles played by mass and by geometry.

[12] Newton's second law is not interpreted by the dynamical explanation's advocates as relating the resultant acceleration to the resultant force as defined by the parallelogram of forces. Doing so would defeat the point of giving a dynamical explanation of the parallelogram of forces. Rather, Newton's second law is interpreted as applicable to each component force separately, as well as to the resultant force regardless of the manner by which component forces produce a resultant. The derivation demonstrates how the resultant force must relate to the forces from which it results.

[13] W. Thomson and P. G. Tait, Ref. 9, pp. 244–245.

[14] Anonymous, "Review of Laplace's Celestial Mechanics," American Quarterly Review **5**, 310–343 (1829).

[15] This equivalence presupposes that equal and opposite forces have zero resultant and that the resultant of two forces is independent of which other forces may also be acting.

[16] J. Challis, *Remarks on the Cambridge Mathematical Studies and Their Relation to Modern Physical Science* (Deighton Bell, Cambridge, 1875), p. 6.

[17] W. Whewell, History of Scientific Ideas Vol. 1, *Being the First Part of the Philosophy of the Inductive Sciences*, 3rd ed., (J. W. Parker, London, 1858), p. 225.

[18] Reference 17, p. 226.

[19] W. H. Macaulay, "The laws of dynamics, and their treatment in textbooks (continued)," Math. Gaz. **1**, 399–404 (1900) The same argument is made in J. Robison, "Dynamics," *Supplement to the Encyclopaedia or Dictionary of Arts, Sciences, and Miscellaneous Literature* (Thomas Dobson, Philadelphia, 1803), Vol. 1, pp. 581–629; J. L. Weisbach, *Theoretical Mechanics*, Vol. 1, translated by E. B. Coxe (Van Nostrand, New York, 1875), p. 178; W. E. Johnson, "Proof of the parallelogram of forces," Nature (London) **41**, p. 153 (1889); A. G. G., "Our bookshelf," *ibid.* **42**, p. 413 (1890); F. R. Moulton, *An Introduction to Celestial Mechanics*, 2nd rev. ed. (Macmillan, New York, 1914), p. 6.

[20] A. H., "Review of 'Whewell's Mechanics—Last Edition'," The Mechan-

ics' Magazine **48**, 103–107 (1848).

[21] See, for instance, J. Cockle, "Notes bearing on mathematical history," Memoires of the Literary and Philosophical Society of Manchester Ser. 3, **6**, 10–15 (1879); E. J. Routh, *A Treatise on Analytical Statics*, 2nd ed. (Cambridge U. P., Cambridge, 1896), Vol. 1, p. 18.

[22] J. L. Lagrange, *Théorie des Fonctions Analytiques*, 2nd ed. (Paris, Courcier, 1813) in *Oeuvres de Lagrange* (Gauther-Villars, Paris, 1881), Vol. 9, pp. 349–350.

[23] W. Browne, *The Student's Mechanics* (C. Griffin, London, 1883), p. 36; cf. H. Goodwin, *An Elementary Course of Mathematics* (Deighton, Cambridge, 1846), p. 272.

[24] Among the texts endorsing this explanation are T. Tate, *The Principles of Mechanical Philosophy* (Longmans, Brown, Green, and Longmans, London, 1853), pp. 72–74; J. Galbraith and S. Haughton, *Manual of Mechanics* (Longmans, Brown, Green, Longmans, Roberts, London, 1860), pp. 7–13; J. A. Skertchly, *Natural Philosophy, Part I. Mechanics* (Thomas Murby, London, 1873), pp. 23–25; T. M. Goodeve, *Principles of Mechanics* (Longmans, Green, London, 1874), pp. 65–68; I. Todhunter, *Mechanics for Beginners* (Macmillan, London, 1878), pp. 19–22.

[25] C. D. M. B. Duchayla, Ref. 4.

[26] A. De Morgan, "Duchayla," Notes and Queries Ser. 3 **5** (130), 527–528 (1864); T. T. Wilkinson, "Duchayla," *ibid.*, **6** (132), p. 39 (1864); I. Grattan-Guinness, *Convolutions in French Mathematics, 1800–1840* (Birkhaüser, Basel, 1990), Vol. 1, p. 310.

[27] I. P. Church, *Mechanics of Engineering* (John Wiley & Sons, New York, 1893), pp. 4–6. Church was a professor of civil engineering at Cornell.

[28] The converse of the transmissibility principle says that if a force's effect would be unaltered if its point of application were changed to another point rigidly connected to its actual point of application, then the line through that other point and the force's actual point of application must lie along the force's direction.

[29] This step will be discussed further in the next section.

[30] Whewell's explanation of the parallelogram law is similar to Duchayla's, although it does not proceed inductively and does not use the transmissibility principle. Instead it uses the principle that on any lever, two forces tending to turn it oppositely balance exactly when they have equal moments about the fulcrum. See W. Whewell, *An Elementary Treatise on Mechanics*, 7th ed. (Deighton's, Cambridge, 1847), pp. 30–32.

[31] W. Mitchell, J. R. Young, and J. Imray, *The Circle of the Sciences, Volume IX: Mechanical Philosophy* (Richard Griffin, London and Glasgow, 1860), p. 47; cf. J. Galbraith and S. Haughton, Ref. 24, p. 7.

[32] See, for instance, H. Goodwin, "On the connection between the sciences of mechanics and geometry," Trans. Cambridge Philos. Soc. **8**, 269–277 (1849); W. H. Besant, "The teaching of elementary mathematics," read at the Annual Meeting of the Association for the Improvement of Geometrical Teaching, January 1883, reported by R. T. in Nature (London) **27**, 581–583 (1883);J. B. Lock, Ref. 9, p. 155; A. G. G., Ref. 19; O. Heaviside, *Electromagnetic Theory* (The Electrician, London, 1893), Vol. 1, p. 147.

[33] W. H. Macaulay, Ref. 19, p. 403.

[34] W. E. Johnson, Ref. 19, p. 153; cf. S. Earnshaw, *A Treatise on Statics*, 3rd ed. (Cambridge U. P., Cambridge, 1845), p. v; Anonymous "On the composition and resolution of forces," The English Journal of Education, n.s., **4**, 378–381 (1850).

[35] J. Challis, *Notes on the Principles of Pure and Applied Calculation* (Deighton, Bell, Cambridge, 1869), p. 98.

[36] W. Thomson and P. G. Tait, Ref. 9, p. 244.

[37] See Ref. 13.

[38] H. Goodwin, Ref. 32, p. 273.

[39] S. D. Poisson, Ref. 5.

[40] Among texts endorsing Poisson's explanation are Anonymous, Ref. 14, p. 314; J. R. Young, *Elements of Mechanics* (Carey, Lea & Blanchard, Philadelphia, 1834), pp. 250–252; S. Earnshaw, Ref. 34, pp. v, 214–220; P. Barlow, "Mechanics," *Encyclopaedia Metropolitana* (Griffin, London, 1848), Vol. 3, pp. 10–11; B. Price, *A Treatise on Infinitesimal Calculus, Vol. 3: A Treatise of Analytical Mechanics*, 2nd ed. (Clarendon Press, Oxford, 1868), pp. 19–21; W. E.Johnson, Ref. 19; W. H.Macaulay, Ref. 19; F. R. Moulton, Ref. 19, p. 403.

[41] For example, J. R. Young, Ref. 40, p. 250 (for Fig. 4), p. 20 (Fig. 5), p. 21 (Fig. 6).

[42] Strictly speaking, Poisson is entitled to conclude only that the resultant lies along the bisector of the larger angle or along the bisector of the smaller angle between P_1 and P_2. To explain why it lies along the smaller angle's bisector, Poisson could have appealed to two further premises: That the resultant of two equal forces depends continuously on their angle and that the resultant of two forces in the same direction points in that direction, too, and thus bisects the smaller angle (measuring zero radians) between them. Suppose that for some angle measure $\beta > 0$, the resultant of two equal forces subtending an angle of measure β bisects the larger angle between them. By continuity, for some angle γ, where $0 < \gamma < \beta$, the resultant of two equal forces subtending an angle of measure γ has zero resultant. By Poisson's further premise that two forces have zero resultant only if they are equal and opposite, it follows that P_1 and P_2 are opposite—a contradiction. Hence, the resultant of two equal forces must bisect the smaller angle between them.

[43] Poisson, following Foncenex and D'Alembert, appealed explicitly to dimensional homogeneity but did not specify that this step presupposes f's continuity. For more on the role of dimensional reasoning in some scientific explanations, see M. Lange, "Dimensional explanations," Nous **43**, 742–775 (2009).

[44] I slightly simplified this step of Poisson's argument, following J. R. Young, Ref. 40, p. 252.

[45] The core of this argument appears in P. Barlow, Ref. 40, p. 9.

[46] A. DeMorgan, "On the general principles of which the composition or aggregation of forces is a consequence," Trans. Cambridge Philos. Soc. **10**, ii, 290–304 (1859).

[47] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon Press, Oxford, 1873), Vol. 1, p. 10; cf. Anonymous, "Composition," *The Penny Cyclopaedia of the Society for the Diffusion of Useful Knowledge* (Charles Knight, London, 1837), Vol. 7, p. 43.

[48] Not every physical quantity with a magnitude and direction composes vectorially. For example, electric current (as used in characterizing an electric circuit) is not a vector quantity despite having a magnitude and being associated with a direction. If two wires are connected to a given pole of a battery, one directed upward and one directed to the right, then it is not true that the net current flow is along the diagonal between them.

[49] See Ref. 33.

[50] Ref. 46, p. 299.

[51] O. Gregory, *A Treatise of Mechanics*, 3rd ed. (FC & J Rivington, London, 1826), Vol. 1, pp. 14–15.

[52] D. Bernoulli, "Examen principiorum mechanicae, et demonstrations geometricae de compositione et resolutione virium," Commentarii Academiae Scientarum Imperialis Petropolitanae **1**, 126–142 (1726) [reprinted in *Die Werke von Daniel Bernoulli*, Band 3, edited by L. P. Bouckaert, D. Speiser, and B. L. van der Waerden (Birkhäuser, Basel, 1982), pp. 119–135]. Forces and accelerations cannot both combine vectorially if any force is proportional to, for example, the square of the acceleration for which it is responsible. Nevertheless, with forces and accelerations each combining vectorially, the fundamental dynamical law could have been that force is proportional to acceleration squared as long as this law applies not to each component separately, but only to the resultant acceleration produced by the resultant force.

[53] J. Cockle, Ref. 21, pp. 12–13.

[54] L. Poinsot, *The Elements of Statics, Part 1*, translated by T. Sutton (Cambridge U. P., Cambridge, 1847), pp. 2–3.

[55] S. Earnshaw, Ref. 34, p. v.

[56] That is, separate and unattached.

[57] Analogous counterfactuals underwrite the power of symmetry principles to explain why conservation laws hold but not vice versa. For further discussion, see M. Lange, "Laws and meta-laws of nature: Conservation laws and symmetries," Stud. Hist. Philos. Mod. Phys. **38**, 457–481 (2007).

[58] A. Eddington, *The Nature of the Physical World* (Macmillan, New York, 1928), pp. 266–267. For more on the relation between counterfactuals and laws of nature, see M. Lange, *Laws and Lawmakers* (Oxford U. P., New York, 2009).